Problem 16

The Buffon Needle Problem (1777)

Problem. *A needle of length l is thrown at random on a plane on which a set of parallel lines separated by a distance d (>l) have been drawn. What is the probability that the needle will intersect one of the lines?*

Solution. Consider the diagram in Fig 16.1. Let the distance between the midpoint of the needle to the nearest line be Y ($0 < Y < d/2$), and the acute angle between the needle and the horizontal be Φ ($0 < \Phi < \pi/2$). We assume that $Y \sim U(0, d/2)$ and that $\Phi \sim U(0, \pi/2)$. It is also reasonable to assume that Y and Φ are statistically independent. Therefore, the joint density of Y and Φ is

$$\begin{aligned} f_{Y\Phi}(y, \varphi) &= f_Y(y) f_\Phi(\varphi) \\ &= \frac{2}{d} \cdot \frac{2}{\pi} \\ &= \frac{4}{\pi d}, \quad 0 < y < d/2,\ 0 < \varphi < \pi/2. \end{aligned}$$

The needle will intersect one of the lines if and only if $Y < (l/2)\sin\Phi$ (see Fig. 16.1). The probability of this event is the double integral of the joint density $f_{Y\Phi}$ over the area A (see Fig. 16.2):

$$\begin{aligned} \Pr\left\{Y < \frac{l}{2}\sin\Phi\right\} &= \iint_A f_{Y\Phi}(y, \varphi)\,dy\,d\varphi \\ &= \int_0^{\pi/2} \left\{ \int_0^{(l\sin\varphi)/2} \frac{4}{\pi d}\,dy \right\} d\varphi \\ &= \frac{2l}{\pi d}. \end{aligned} \tag{16.1}$$

Hence the required probability is $(2l)/(\pi d)$.

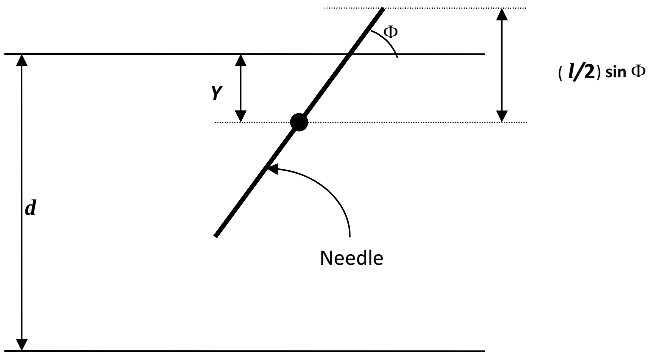

Figure 16.1 The Buffon Needle Problem.

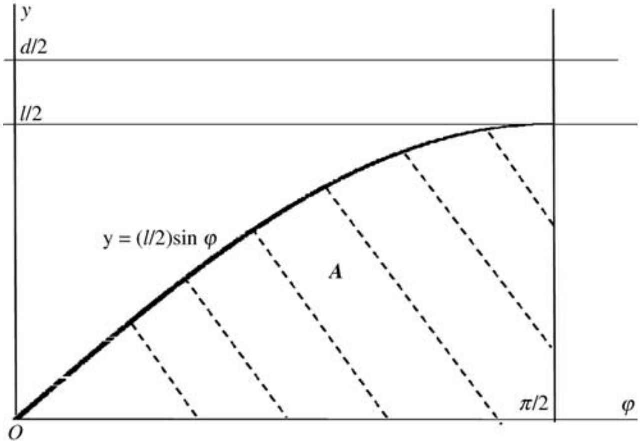

Figure 16.2 Graph of $y = (l/2)\sin\varphi$ vs. φ ($d > l$ case).

16.1 Discussion

A simpler solution than the previous can be obtained by first noting that (Y, Φ) is uniform on $S = \{(y, \varphi) : 0 < y < d/2, 0 < \varphi < \pi/2\}$. Therefore the required probability is a "geometric" probability, namely the ratio of the area of A to that of S (see Fig. 16.2):†

$$\begin{aligned}\text{Prob. of intersection} &= \frac{\int_0^{\pi/2}(l/2)\sin\varphi\, d\varphi}{(\pi/2)(d/2)} \\ &= \frac{2l}{\pi d}\cdot[-\cos\varphi]_{\varphi=0}^{\pi/2} \\ &= \frac{2l}{\pi d},\end{aligned}$$

as we obtained before in Eq. (16.1).

† In general, suppose a region G contains a smaller region g. Then if a point is chosen at random in G, the (geometric) probability that the point lies within g is given by the ratio of the measure (e.g. volume, area or length) of g to that of G.

Figure 16.3 Georges-Louis Leclerc, Comte de Buffon (1707–1788).

The *Buffon Needle Problem*[†] is considered to be the first problem in geometric probability, which is an extension of the mathematical definition of probability. It was enunciated in 1733 by Georges-Louis Leclerc, Comte de Buffon (1707–1788), French naturalist and mathematician (Fig. 16.3). The problem was afterward published with the correct solution in Buffon's article *Essai d'Arithmétique Morale* (Buffon 1777, p. 96) (see Fig. 16.4). Laplace later observed that the formula $p = 2l/(\pi d)$ (where p is the probability of intersection) could actually be used to estimate the value of π (Laplace, 1812, p. 360). If a needle is thrown N times and in n cases the needle intersects one of the lines, then n/N is an unbiased estimator of p. An estimator of π is therefore $(2lN)/(nd)$. However, this estimator is *biased*[‡] because $(nd)/(2lN)$ is an *unbiased* estimator of $1/\pi$.

[†] The Buffon needle problem is also discussed in Solomon (1978, p. 2), van Fraassen (1989, p. 303), Aigner and Ziegler (2003, Chapter 21), Higgins (2008, p. 159), Beckmann (1971, p. 159), Mosteller (1987, p. 14), Klain and Rota (1997, Chapter 1), Lange (2010, p. 28), and Deep (2006, p. 74).

[‡] If T is used as an estimator of a parameter t, then the bias of T is defined as $\text{Bias}_t(T) = \mathscr{E}T - t$.

Dans une chambre parquetée ou pavée de carreaux égaux, d'une figure quelconque, on jette en l'air un écu; l'un des joueurs parie que cet écu après ſa chute ſe trouvera à franc-carreau, c'eſt-à-dire, ſur un ſeul carreau; le ſecond parie que cet écu ſe trouvera ſur deux carreaux, c'eſt-à-dire, qu'il couvrira un des joints qui les ſéparent; un troiſième joueur parie que l'écu ſe trouvera ſur deux joints; un quatrième parie que l'écu ſe trouvera ſur trois, quatre ou ſix joints: on demande les ſorts de chacun de ces joueurs.

Je cherche d'abord le ſort du premier joueur & du ſecond; pour le trouver, j'inſcris dans l'un des carreaux une figure ſemblable, éloignée des côtés du carreau, de la longueur du demi-diamètre de l'écu; le ſort du premier joueur ſera à celui du ſecond, comme la ſuperficie de la couronne circonſcrite eſt à la ſuperficie de la figure inſcrite; cela peut ſe démontrer aiſément, car tant que le centre de l'écu eſt dans la figure inſcrite, cet écu ne peut être que ſur un ſeul carreau, puiſque par conſtruction cette figure inſcrite eſt par-tout éloignée du contour du carreau, d'une diſtance égale au rayon de l'écu; & au contraire dès que le centre de l'écu tombe au dehors de la figure inſcrite, l'écu eſt néceſſairement ſur deux ou pluſieurs carreaux, puiſqu'alors ſon rayon eſt plus grand que la diſtance du contour de cette figure inſcrite au contour du carreau; or, tous les points où peut tomber

Figure 16.4 Extract from Buffon's Needle Problem, taken from Buffon's *Essai d'Arithmétique Morale* (Buffon, 1777, p. 96).

Table 16.1[†] lists several attempts to estimate the value of π. The one by Lazzarini (1902) is of special interest since it is accurate up to six places of decimal to the true value of π.[‡] Using reasonable assumptions, Coolidge (1925, p. 82) shows that the probability of attaining such an accuracy by chance alone is about .014 and concludes

> It is much to be feared that in performing this experiment Lazzarini 'watched his step'.

Indeed, as Kendall and Moran (1963, p. 71) point out, the accurate approximations in Table 16.1 suggest that "optional stopping" was used.[§]

[†] Taken from Kendall and Moran (1963, p. 70).

[‡] And in fact coincides with 355/113, which is a more accurate value than 22/7 to approximate π and which was first discovered by the Chinese mathematician Zu Chongzhi (429–500). See Posamentier and Lehman (2004, p. 61) for more details.

[§] For a fuller analysis of most of these experiments, see Gridgeman (1960) and O'Beirne (1965, pp. 192–197).

Table 16.1 Attempts to Estimate π Using Eq. (16.1)

Experimenter	Needle length	No. of throws	No. of hits	Estimate of π
Wolf (1850)	.8	5000	2532	3.1596
Smith (1855)	.6	3204	1218.5	3.1553
De Morgan (1860)	1.0	600	382.5	3.137
Fox (1884)	.75	1030	489	3.1595
Lazzarini (1902)	.83	3408	1808	3.1415929
Reina (1925)	.5419	2520	859	3.1795

We now present an intriguing alternative solution (e.g., see Uspensky, 1937, p. 253) to the *Buffon Needle Problem*, due to the French mathematician Joseph Emile Barbier (1839–1889) (Barbier, 1860). The peculiarities of the approach are that it does not involve any integration and can be generalized to arbitrary convex shapes. First we write $p \equiv p(l)$to emphasize that the probability of intersection depends on the length of the needle. Suppose the needle is divided into two parts with lengths l' and l''. Since a line intersects the needle if and only if it intersects either portion, we have $p(l) = p(l') + p(l'')$. The latter equation is satisfied if

$$p(l) = kl, \tag{16.2}$$

where k is a constant of proportionality. Now imagine a polygonal line (not necessarily convex) of total length l made up of n rectilinear segments $a_1, a_2, \ldots, a_n$ where each $a_j < d$. Each segment a_j has a probability $p(j) = ka_j$ of intersecting one of the parallel lines. Let

$$I_j = \begin{cases} 1, & \text{if segment } a_j \text{ intersects the line,} \\ 0, & \text{otherwise.} \end{cases}$$

The total number of intersections is then

$$T = \sum_{j=1}^{n} I_j,$$

and the expectation of T is

$$\begin{aligned} \mathscr{E}T &= \sum_{j=1}^{n} \mathscr{E}\, I_j \\ &= \sum_{j=1}^{n} ka_j \\ &= kl. \end{aligned} \tag{16.3}$$

For a circle with diameter d, we have $T = 2$, $\mathscr{E}T = 2$, and $l = \pi d$. Using Eq. (16.3), we solve for k and obtain $k = 2/(\pi d)$. Hence, Eq. (16.2) becomes

$$p(l) = kl = \frac{2l}{\pi d},$$

which is the same formula we obtained using integrals in Eq. (16.1).

Digging deeper, if we consider a sufficiently small convex polygon with circumference C, then we can have only two intersections (with probability, say P) or zero intersection (with probability $1-P$). Then the expected number of intersections is $2P$. From Eq. (16.3), we have $\mathscr{E}T = kl$, where $k = 2/(\pi d)$ as we showed previously. Substituting $\mathscr{E}T = 2P$, $k = 2/(\pi d)$, and $l = C$ in $\mathscr{E}T = kl$, we have

$$P = \frac{C}{\pi d}. \tag{16.4}$$

This gives a general formula for the probability P of intersection of a small convex polygon (of any shape) with circumference C when thrown on lines separated by a distance d. The formula is interesting because P does not depend either on the number of sides or on the lengths of the sides of the polygon. It therefore also applies to any convex contour. For an alternative derivation of Eq. (16.4), see Gnedenko (1978, p. 39).

Two variations of the classic *Buffon Needle Problem* are also worth considering. First

> *A needle of length l is thrown at random on a plane on which a set of parallel lines separated by a distance d, where d < l. What is the probability that the needle will intersect one of the lines?*

This time, the length of the needle is greater than the separation of the lines but (Y, Φ) is still uniform on $S = \{(y, \varphi) : 0 < y < d/2,\ 0 < \varphi < \pi/2\}$. Therefore the required probability, as a geometric probability, is the ratio of the area of A' to that of S (see Fig 16.5)[†]:

$$\begin{aligned}
\Pr\left\{Y < \frac{l}{2}\sin\Phi\right\} &= \frac{\displaystyle\int_0^{\arcsin(d/l)} \frac{l}{2}\sin\varphi d\varphi + \left(\frac{\pi}{2} - \arcsin\frac{d}{l}\right)\frac{d}{2}}{\left(\dfrac{\pi}{2}\right)\left(\dfrac{d}{2}\right)} \\
&= \frac{4}{\pi d}\left\{-\frac{l}{2}\left[\cos\varphi\right]_{\varphi=0}^{\arcsin(d/l)} + \frac{d}{2}\arccos\left(\frac{d}{l}\right)\right\} \\
&= \frac{4}{\pi d}\left[-\frac{l}{2}\left\{\cos\left(\arcsin\frac{d}{l}\right) - 1\right\}\right] + \frac{2}{\pi}\arccos\left(\frac{d}{l}\right) \\
&= \frac{2l}{\pi d}\left(1 - \sqrt{1 - \frac{d^2}{l^2}}\right) + \frac{2}{\pi}\arccos\left(\frac{d}{l}\right).
\end{aligned} \tag{16.5}$$

[†] In the following, we will use $\arcsin(x) + \arccos(x) = \pi/2$ and $\cos(\arcsin(x)) = \sqrt{1-x^2}$.

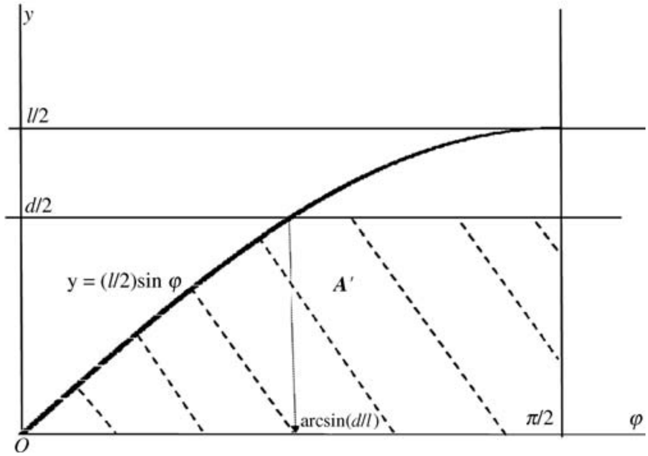

Figure 16.5 Graph of $y = (l/2)\sin\varphi$ vs. φ ($d < l$ case).

The second variation was first considered by Laplace† (1812, pp. 360–362) and is as follows (Fig. 16.6):

> *Suppose a needle of length l is thrown on a plane that is made up of congruent rectangles each of dimensions c and d (l < c, d). What is the probability that the needle will intersect at least one of the sides of the rectangle?*

Let the distances from the base of the needle to the next vertical line along the horizontal axis be X and to the next horizontal line along the vertical axis be Y. Let the angle made by the length of the needle with the horizontal be Φ. We assume that X, Y, and Φ are uniformly distributed on $(0, c)$, $(0, d)$, and $(0, \pi/2)$, respectively. The needle will cross a vertical line if $X < l\cos\Phi$. The probability of this event is again calculated through geometric probability:

$$p_V = \frac{\int_0^d \left\{\int_0^{\pi/2} l\cos\varphi\, d\varphi\right\} dy}{\pi c d/2} = \frac{2l}{\pi c}.$$

Similarly, the needle will cross a horizontal line if $Y < l\sin\Phi$, and this will happen with probability

$$p_H = \frac{\int_0^c \left\{\int_0^{\pi/2} l\sin\varphi\, d\varphi\right\} dx}{\pi c d/2} = \frac{2l}{\pi d}.$$

Both horizontal and vertical intersections occur if $X < l\cos\Phi$ and $Y < l\sin\Phi$, and the probability is

$$p_{V\times H} = \frac{\int_0^{(\pi/2)} \int_0^{(l\sin\varphi)} \int_0^{(l\cos\varphi)} dx\,dy\,d\varphi}{\pi c d/2} = \frac{l^2}{\pi c d}.$$

† See also Arnow (1994), Solomon (1978, p. 3), and Uspensky (1937, p. 256).

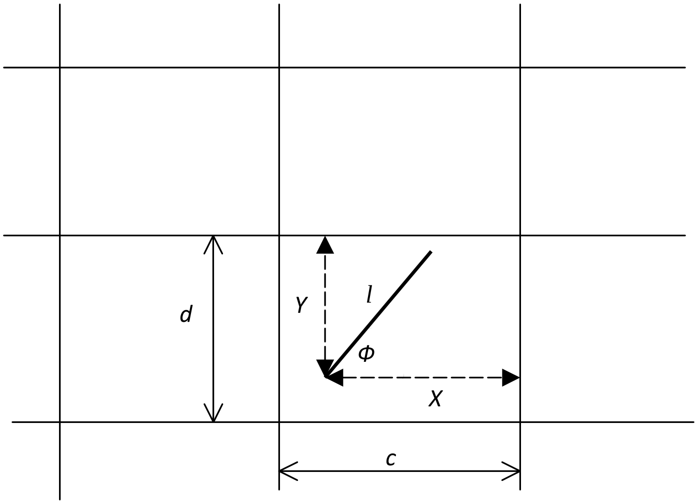

Figure 16.6 Laplace's extension of the Buffon Needle Problem.

Hence, the probability of either a vertical or horizontal intersection is

$$\begin{aligned} p_{V+H} &= p_V + p_H - p_{V\times H} \\ &= \frac{2l}{\pi c} + \frac{2l}{\pi d} - \frac{l^2}{\pi cd} \\ &= \frac{2l(c+d) - l^2}{\pi cd}. \end{aligned} \tag{16.6}$$

Note that, as $c \to \infty$, we have

$$\lim_{c\to\infty} p_{V+H} = \lim_{c\to\infty} \frac{2l(c+d) - l^2}{\pi cd} = \frac{2l}{\pi d},$$

which is the same probability as in Eq. (16.1).

We now consider the following issue. Out of the several possible uniform random variables in the *Buffon Needle Problem*, why did we choose the acute angle between the needle and the nearest line (Φ), and the distance between the midpoint of the needle to the nearest line (Y)? Would the probability of intersection have changed if we had instead chosen the vertical rise of the needle from its midpoint and the distance of the midpoint to the nearest line as uniformly distributed each, that is, $\Phi' \sim U(0, l/2)$ and $Y \sim U(0, d/2)$, where $\Phi' = (l \sin \Phi)/2$? The answer is yes. A similar calculation to the one presented in the solution shows that the probability of intersection would then be $l/(2d)$, which is different from Eq. (16.1). This might seem quite disquieting. However, as van Fraassen (1989, p. 312) has shown, $\Phi \sim U(0, \pi/2)$ and $Y \sim U(0, d/2)$ are choices that make the joint density of (Y, Φ) invariant under rigid motions. That is, if the lines were to be rotated or translated before the needle was thrown on them, these choices ensure we get the same probability of intersection.

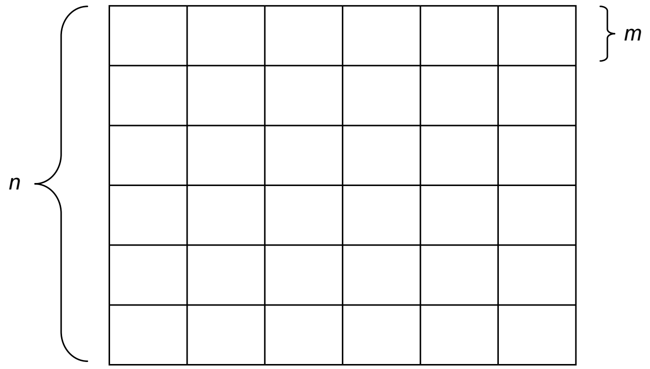

Figure 16.7 Procedure to estimate e geometrically. The big square is of dimension $n \times n$ and each of the small squares is of dimension $m \times m$ (n/m is assumed to be an integer).

The invariance of this probability under such translations seems to be a natural desideratum.†

Since we have shown how the value of π could be estimated through geometric probability (see first paragraph in Discussion section), we might ask a related question:

Can we also use a geometric procedure to estimate the exponential number e?

Nahin (2000, p. 30) provides a geometric approach,‡ which we adapt in the following. Imagine a big square of dimension $n \times n$ made up of identical smaller squares each of dimension $m \times m$, as shown in Fig. 16.7. Let us choose m so that n/m is an integer. The number of small squares is thus $N = n^2/m^2$.

Now, suppose N darts are thrown at random on the big square. Let X be the number of darts that land inside a particular small square. Then $X \sim \mathrm{B}(N, 1/N)$ and

$$\Pr\{X = x\} = \binom{N}{x}\left(\frac{1}{N}\right)^x \left(1 - \frac{1}{N}\right)^{N-x}, \quad x = 0, 1, 2, \ldots, N.$$

The probability that the small square receives no dart is

$$\Pr\{X = 0\} = \left(1 - \frac{1}{N}\right)^N \rightarrow e^{-1} \quad \text{as } N \rightarrow \infty.^{\S}$$

We thus have a method to estimate e. Let us make m small enough so that N is large. Suppose, after the N darts have been thrown, the number of small squares (out of the N) that receive no dart at all is s. Then $s/N \approx 1/e$ so that $e \approx N/s$. Again, N/s is a biased estimator of e since s/N is unbiased for $1/e$.

† See **Problem 19** for more on invariance.

‡ Of course, if one is not interested in a geometric approach one can simply use $e = 1 + 1/1! + 1/2! + 1/3! + \cdots$.

§ Using the fact that $\lim_{x \rightarrow \infty}(1 + a/x)^x = e^a$.

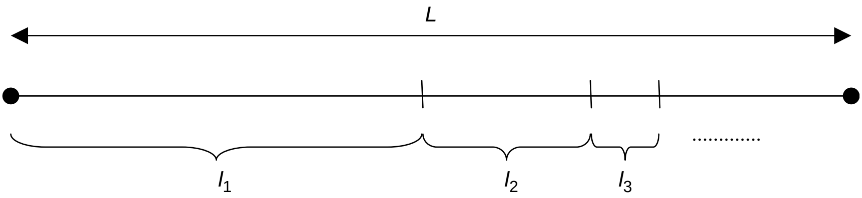

Figure 16.8 Geometric unbiased estimation of e.

There exists a second geometric approach† that can give an *unbiased* estimate of e. Suppose a line of length L is divided into successively smaller segments $I_n = \left(\frac{n-1}{n}L, \frac{n}{n+1}L\right]$ for $n = 1, 2, \ldots$ (see Fig. 16.8).

A sharp knife is then thrown at random perpendicular to the line and if it cuts the line across the jth segment, we then assign the random variable X the value $2 + 1/(j-1)!$ We now show that $\mathscr{E}X = e$. We have

$$\Pr\{X = x\} = \begin{cases} \dfrac{j}{j+1} - \dfrac{j-1}{j} = \dfrac{1}{j(j+1)}, & \text{for } x = 2 + 1/(j-1)!, \ j = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\begin{aligned} \mathscr{E}X &= \sum_{j=1}^{\infty}\left\{2 + \frac{1}{(j-1)!}\right\} \cdot \frac{1}{j(j+1)} \\ &= \sum_{j=1}^{\infty}\left\{\frac{2}{j} - \frac{2}{(j+1)} + \frac{1}{(j+1)!}\right\} \\ &= \left(\frac{2}{1} - \frac{2}{2} + \frac{2}{2} - \frac{2}{3} + \frac{2}{3} - \frac{2}{4} + \cdots\right) + \sum_{j=1}^{\infty}\frac{1}{(j+1)!} \\ &= 2 + \sum_{j=1}^{\infty}\frac{1}{(j+1)!} \\ &= \sum_{j=0}^{\infty}\frac{1}{j!} \\ &= e. \end{aligned}$$

Thus to obtain an unbiased estimate of e, we throw the knife on the line n times (where n is large), note the segment cut each time and record the corresponding values $x_1, x_2, \ldots, x_n$ of X. Then an unbiased estimate of e is $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$.

† This method was communicated to me by Bruce Levin.